THE

TRANSACTIONS

OF THE

ROYAL IRISH ACADEMY.

VOLUME XXXI.—PART V.

THE TWELFTH AND CONCLUDING MEMOIR ON THE "THEORY OF SCREWS," WITH A SUMMARY OF THE TWELVE MEMOIRS.

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DUBLIN:
PUBLISHED AT THE ACADEMY HOUSE, 19, DAWSON STREET.
SOLD ALSO BY
HODGES, FIGGIS, & CO. (LIMITED), 104, GRAFTON STREET.
AND BY WILLIAMS & NORGATE,
14, HENRIETTA STREET, COVENT GARDEN, LONDON;
20, SOUTH FREDERICK STREET, EDINBURGH;
AND 7, BROAD STREET, OXFORD.
1898.

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V.

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[Read November 8, 1897.]

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Introductory Note.

The Academy has done me the honour of publishing in these Transactions a long series of memoirs on the "Theory of Screws," to which must be added one in the Philosophical Transactions.* These memoirs examined various portions of an extensive subject. There was, however, a vital part of the geometrical development which, for more than twenty years, I had striven in vain to effect.

Within the last few months I have overcome my long standing difficulties. The results are here given in the Twelfth and concluding Memoir of the series.

* See the "Summary," p. 185.

I.—Preliminary.

It will be remembered how Poinsot advanced our knowledge of the dynamics of a rigid system by a beautiful geometrical theory of the rotation of a rigid body about a fixed point. We now specially refer to the geometrical construction by which, when the plane of the instantaneous couple was given, the axis about which the body commenced to rotate was determined.

We may, in successive steps, enunciate with increasing generality the problem which, in its simple form, Poinsot had made classical. From the case of a rigid body which is constrained to rotate around a fixed point we advance to the wider conception of a body which has three degrees of freedom of the most general type. We can generalize this again into the case in which the body, instead of having a definite number of degrees of freedom has any number of such degrees. The range extends from the first or lowest degree, where the only movement of which the body is capable is that of twisting about a single fixed screw, up to the case in which the body being perfectly free, or in other words, having six degrees of freedom, is able to twist about every screw in space. It will, of course, be borne in mind that only small movements are to be understood.

In a corresponding manner we may generalize the forces applied to the body. In the problem of Poinsot already referred to the forces consist solely of a couple. Any other force would be, in this case, destroyed by the reaction of the fixed point. But in the more generalized problems with which the theory of screws is concerned, we do not restrict the forces to the specialized pair which form a couple. We may take a system of forces of the most general type. Such a system, when applied to a rigid body, can be represented by a wrench upon a screw. Thus, by generalizing the freedom of the rigid body, as well as the forces which act upon it, we may investigate the geometrical theory of the motion when a rigid body of the most general type, possessing a certain number of degrees of freedom of the most general type, is disturbed from a position of rest by an impulsive system of forces of the most general type.

II.—One Pair of Impulsive and Instantaneous Screws.

For our present purpose, we may regard a rigid body as completely specified by its three principal axes through the centre of gravity with the three corresponding radii of gyration. There are thus nine coordinates required for the complete specification of the body. The centre of the momental ellipsoid lies at the centre of gravity. The principal axes of the ellipsoid lie along the principal axes of the rigid body, while the lengths of the principal axes of the ellipsoid are inversely proportional to the radii of gyration along the same directions.

Let it be supposed that nothing is known of the position, mass, or other circumstances of an unconstrained rigid body save what can be deduced from the fact that, when struck from a position of rest by an impulsive wrench on a specified screw η, the effect is to make the body commence to move by twisting around a specified screw α.

As α, like every other screw is defined by five coordinates, the knowledge of this screw gives us five data towards the nine data that are required for the complete specification of the rigid body and its position.

We have first to prove that the five elements which can be inferred with respect to the rigid body are—

(1) *A diameter of the momental ellipsoid.*

This gives clearly two data, inasmuch as it restricts the position of the centre of gravity to a determinate ray.

(2) *The radius of gyration about this diameter.*

This is, of course, one datum.

(3) *A ray in the plane conjugate to that diameter.*

A point in the plane would have been one datum, but a ray in the plane is two. If the centre of gravity were also known, we should at once be able to draw the conjugate plane.

Draw a plane through both the instantaneous screw α and the common perpendicular to α and η. It can be easily proved by elementary considerations, that the centre of gravity of the rigid body must lie in that

plane. It can also be shown that if p_α be the pitch of α, and if $\hat{\alpha\eta}$ represent the angle between α and η, then the perpendicular distance of of the centre of gravity from α will be expressed by $p_\alpha \tan \hat{\alpha\eta}$. This expression is completely known since α and η are known. Thus we find that the centre of gravity must lie in a determinate ray parallel to α. There is no ambiguity as to the side on which this ray lies when it is observed that it must pass between α and the point in which η is met by the common perpendicular to η and α. Thus from knowing α and η we discover a diameter of the momental ellipsoid.

If $\dot{\alpha}$ be the twist velocity with which a rigid body of mass M is twisting about any screw α. If η be the corresponding impulsive screw, and if $\varpi_{\alpha\eta}$ denote as usual the virtual coefficient of α and η; then it is proved in the Eleventh Memoir that the kinetic energy of the body admits of the concise expression

$$M\dot{\alpha}^2 \frac{p_\alpha}{\cos \hat{\alpha\eta}} \varpi_{\alpha\eta}.$$

We make use of this to obtain the value of ρ_α^2 where ρ_α is the radius of gyration about an axis parallel to α through the centre of gravity. The kinetic energy may also be expressed thus—

$$\tfrac{1}{2} M\dot{\alpha}^2 (\rho_\alpha^2 + p_\alpha^2 + \alpha_\alpha^2).$$

By equating these expressions we find

$$\rho_\alpha^2 = 2 \frac{p_\alpha}{\cos \hat{\alpha\eta}} \varpi_{\alpha\eta} - p_\alpha^2 - \alpha_\alpha^2.$$

But when α and η are known, the three terms on the right-hand side of this equation are determined. Thus we learn the radius of gyration on the diameter parallel to α.

It remains to show how the ray in the conjugate plane is determined. This appears as follows:—Let a screw θ, of zero pitch, be placed on that known diameter of the momental ellipsoid which is parallel to α. Draw a cylindroid through the two screws θ and η. Let ϕ be the other screw of zero pitch on the same cylindroid. There must always be such a screw,

because the cylindroid has two screws of zero pitch, and one of these being real by hypothesis the other must be real also.

We could replace the original impulsive wrench on η by its two component wrenches on any two screws of the cylindroid. We choose for this purpose the two screws of zero pitch θ and ϕ. Thus we replace the wrench on η by two forces, whose joint effect is identical with the effect that would have been produced by the wrench on η.

As to the force along the line θ it is, from the nature of the construction, directed through the centre of gravity. Such an impulsive force would produce a velocity of translation, but it could have no effect in producing a rotation. The rotatory part of the initial twist velocity must therefore be solely the result of the impulsive force on ϕ.

But when an impulsive force is applied to a quiescent rigid body we know, from Poinsot's theorem, that the rotatory part of the instantaneous movement must be about an axis parallel to the direction which is conjugate in the momental ellipsoid to the plane which contains both the centre of gravity and the impulsive force. It follows that the ray ϕ must be situated in that plane which is conjugate in the momental ellipsoid to the diameter parallel to α. But as we have already seen, the position of ϕ is completely defined on the known cylindroid on which it lies. We have thus obtained a fixed ray in the conjugate plane to a known diameter of the momental ellipsoid.

The three statements at the beginning of this section have therefore been established. We have, accordingly, ascertained five distinct geometrical data towards the complete specification of the rigid body in its abstract form. These five data are inferred solely from information conveyed by the circumstance that we are given a single pair of corresponding impulsive and instantaneous screws.

III.—Two and three pairs of Impulsive and Instantaneous Screws.

Let us next suppose that we are given a second pair of corresponding impulsive and instantaneous screws. We shall examine how much further we are enabled to proceed by the help of this additional information

towards the complete determination of the rigid body in its abstract form. What is here set down is to be regarded as the extension and completion of what was said on the same subject in the Eleventh Memoir.

We have already seen that for the full specification of the rigid body in its abstract form just nine data are necessary. Less than nine would be inadequate. Any data in excess of nine, if not actually impossible, would be superfluous. If, therefore, we be given a second pair of impulsive and instantaneous screws, the five data which they bring cannot be wholly independent of the five data brought by the preceding pair. It is therefore plain that the quartet of screws forming two pairs of corresponding impulsive screws and instantaneous screws cannot be chosen arbitrarily. They must submit to at least one purely geometrical condition, so that the number of data independent of each other shall not exceed nine.

It is, however, not so obvious, though it is certainly true, that the two pairs of screws must conform not merely to one, but to no less than two geometrical conditions. In fact, if η, ξ be two impulsive screws, and if α, β be the two corresponding instantaneous screws, then, when the body acted upon is perfectly free, the following two formulæ must be satisfied (see Eleventh Memoir):—

$$\frac{p_\alpha}{\cos \hat{\alpha\eta}} \varpi_{\beta\eta} = \frac{p_\beta}{\cos \hat{\beta\xi}} \varpi_{\alpha\xi},$$

$$2\varpi_{\alpha\beta} = \frac{p_\alpha}{\cos \hat{\alpha\eta}} \cos \hat{\beta\eta} + \frac{p_\beta}{\cos \hat{\beta\xi}} \cos \hat{\alpha\xi}.$$

We can determine two geometrical properties of the two pairs of screws which are equivalent to the conditions expressed by these equations.

In the first place, each of the pairs of screws determines a diameter of the momental ellipsoid. The fact that the two diameters, so found, must intersect each other, is obviously one geometrical condition imposed on the system α, η and β, ξ.

Let G be this intersection, and draw GP parallel to α and equal to the radius of gyration about GP, which we have shown to be known from

the fact, that α and η are known. Let X be the plane conjugate to GP in the momental ellipsoid, then this plane is also known.

In like manner, draw GQ parallel to β and equal to the radius of gyration about GQ. Let Y be the plane, conjugate to GQ, in the momental ellipsoid.

Let P_1 and P_2 be the perpendiculars from P, upon X and Y, respectively.

Let Q_1 and Q_2 be the perpendiculars from Q, upon X and Y, respectively.

Then, from the properties of the ellipsoid, it is easily shown that

$$P_1 \cdot P_2 = Q_1 \cdot Q_2.$$

This is the second geometrical relation between the two pairs of screws α, η and β, ξ.

From the fact that these two relations exist, it is evident that the knowledge of a second pair of corresponding impulsive screws and instantaneous screws brings but three independent data. We thus obtain from our knowledge of the two pairs of screws together no more than eight data. We are consequently short by one of the number requisite for the complete specification of the rigid body in its abstract form.

It follows that we must have a singly infinite number of rigid bodies, every one of which will fulfil the necessary conditions. For each and every one of those bodies, α is the instantaneous screw about which twisting motion would be produced by an impulsive wrench on η. For each and every one of those bodies, β is the instantaneous screw about which twisting motion would be produced by an impulsive wrench on ξ. Of course it will be understood that in all cases the four screws submit to the two fundamental geometrical conditions.

We have now to study the geometrical relations of a system of rigid bodies in singly infinite variety which stand to the four screws in the relation just specified.

Draw the cylindroid (α, β) which passes through the two screws α and β. Draw also the cylindroid (η, ξ) which passes through the two corresponding screws η and ξ. It is easily seen that every screw on the

first of these cylindroids if regarded as an instantaneous screw, with respect to the same rigid body, will have its corresponding impulsive screw on the second cylindroid. For any impulsive wrench on (η, ξ) can be decomposed into impulsive wrenches on η and ξ. The first of these will generate a twist velocity about α. The second will generate a twist velocity about β. These two can only compound into a twist velocity about some other screw on the cylindroid (α, β). This must, therefore, be the instantaneous screw corresponding to the original impulsive wrench on (η, ξ).

It is a remarkable point about this part of our subject that, as explained in the Eleventh Memoir, we can now, without any further attention to the rigid body, correllate definitely each of the screws on the instantaneous cylindroid with its correspondent on the impulsive cylindroid.

Select on the cylindroid (α, β) that one screw β_0 which is reciprocal to ξ, and on the cylindroid (η, ξ) that one screw ξ_0 which is reciprocal to β. Then β and β_0 must be conjugate screws of inertia, and consequently ξ_0 is the impulsive screw corresponding to β_0 as an instantaneous screw (First Memoir).

Take any screw γ on the cylindroid (α, ρ) and ascertain the anharmonic ratio of the flat pencil of four rays parallel to α, β, β_0, γ.

Take a screw ζ on the cylindroid (η, ξ) such that the anharmonic ratio of the flat pencil of four rays parallel to η, ξ, ξ_0, ζ shall be equal to the anharmonic ratio on the preceding paragraph.

Then ζ will be the impulsive screw corresponding to γ as the instantaneous screw (Second Memoir).

We thus see how, from our knowledge of two pairs of correspondents, we can construct the impulsive screw on the cylindroid (η, ξ) corresponding to every screw on the cylindroid (α, β).

It has been already explained in the last section, how a single known pair of corresponding impulsive and instantaneous screws suffice to point out a diameter of the momental ellipsoid, and also give its radius of gyration. A second pair of screws will give another diameter of the momental ellipsoid, and these two diameters give, by their intersection, the centre of gravity. As we have an infinite number of corresponding pairs, we thus get an infinite number of diameters, all, however, being

parallel to the principal plane of the instantaneous cylindroid. The radius of gyration on each of these diameters is known. Thus we get a section S of the momental ellipsoid, and we draw any pair of conjugate diameters in that section. These diameters, as well as the radius of gyration on each of them, are thus definitely fixed.

When we had only a single pair of corresponding impulsive and instantaneous screws, we could still determine one ray in the conjugate plane to the diameter parallel to the instantaneous screw. Now that we have further ascertained the centre of gravity, the conjugate plane to the diameter, parallel to the instantaneous axis, is completely determined. Every pair of corresponding impulsive and instantaneous screws will give a conjugate plane to the diameter parallel to the instantaneous screw. Thus we know the conjugate planes to all the diameters in the plane S. All these planes must intersect, in a common ray Q, which is, of course, the conjugate direction to the plane S.

This ray Q might have been otherwise determined. Take one of the two screws, of zero pitch, in the impulsive cylindroid (η, ξ). Then the plane, through this screw and the centre of gravity, must, by Poinsot's theorem already referred to, be the conjugate plane to some ray in S. Similarily, the plane through the centre of gravity and the other screw of zero pitch, on the cylindroid (η, ξ), will also be the conjugate plane to some ray in S. Hence, we see, that the ray Q must lie in each of the planes so constructed, and is therefore determined. In fact, it is merely the transversal drawn from the centre of gravity to intersect both the screws of zero pitch on the cylindroid (η, ξ).

We have thus proved that when two pairs of corresponding impulsive screws and instantaneous screws are given, we know the centre of the momental ellipsoid, we know the directions of three of its conjugate diameters, and we know the radii of gyration on two of those diameters. The radius of gyration on the third diameter remains arbitrary. Be that radius what it may, the rigid body will still fulfil the condition rendering α, η and β, ξ respective pairs of instantaneous screws and impulsive screws. We had from the first foreseen that the data would only provide eight coordinates, while the specification of the body

required nine. We now see, precisely, the nature of the undetermined coordinate.

It appears from this investigation that, if two pairs of impulsive screws and the corresponding instantaneous screws are known, but that if there be no other information, the rigid body is indeterminate. It therefore follows that, if an impulsive screw be given, the corresponding instantaneous screw will not generally be determined. Each of the possible rigid bodies will have a different instantaneous screw, though the impulsive screw may be the same. It was, however, shown, in the Eleventh Memoir, that all the instantaneous screws which pertain to a given impulsive screw lie on the same cylindroid. It is a cylindroid of extreme type, possessing a screw of infinite pitch, and degenerating to a plane, instead of being, as usual, a ruled cubic surface.

Even while the body is thus indeterminate, there are, nevertheless, a system of impulsive screws which have the same instantaneous screw for every rigid body which complies with the expressed conditions. Of course, the several screws on the impulsive cylindroid (η, ξ) have each precisely the same corresponding screw on the instantaneous cylindroid (α, β), whatever may be the body of the system to which the impulsive wrench is applied. But the pairs of screws on these two cylindroids are only an infinitesimal part of the pairs of screws that are circumstanced in this particular way. We have to show that there is a system of screws of the fifth order, such that an impulse delivered by a wrench on any one of those screws η will make any body of the system commence to twist about the same screw α.

As already explained, the system of rigid bodies have a common centre of gravity. Any force, directed through the centre of gravity, will produce a linear velocity parallel to that force. This will, of course, apply to every body of the system. All possible forces, which could be applied to one point, form a system of the third order of a very specialized type. Each one of the screws of this system will have, as its instantaneous screw, a screw of infinite pitch parallel thereto. We have thus a system of impulsive screws of the third order, and a corresponding system of instantaneous screws of the third order, the relation between

each pair being quite independent of whatever particular rigid body of the group the impulsive wrench be applied to.

But this system of the third order is, itself, only a very small part of the system of screws possessing the required property. Take any screw θ, of zero pitch, passing through the centre of gravity, and any screw, ϕ, on the cylindroid (η, ξ). We know, of course, as already explained, the instantaneous screws corresponding to θ and ϕ. Let us call them λ, μ, respectively. Draw the cylindroid (θ, ϕ), and the cylindroid (λ, μ). The latter of these will be the locus of the instantaneous screws, corresponding to the screws on the former as impulsive screws. From the remarkable property of the two cylindroids, so related, it follows that every impulsive screw on (θ, ϕ) will have its corresponding instantaneous screw on (λ, μ) definitely fixed. This will be so, notwithstanding the arbitrary element remaining in the rigid body. From the way in which the cylindroid (θ, ϕ) was constructed, it is plain that the screws belonging to it are members of the system of the fifth order, formed by combinations of screws on the cylindroid (η, ξ) with screws of the special system of the third order passing through the centre of gravity. The system of the fifth order consists of the screws which are reciprocal to that single screw, of zero pitch, which passes through the centre of gravity and cuts the two screws, of zero pitch, on the impulsive cylindroid (η, ξ). The corresponding instantaneous screws will also form a system of the fifth order, but it will be a system of a specialized type. It will be the result of compounding all possible displacements by translation, with all possible twists, about screws on the cylindroid (α, β). The resulting system of the fifth order consists of all screws, of whatsoever pitch, which fulfil the single condition of being perpendicular to the axis of the cylindroid (α, β). Hence we obtain the following theorem :—

If an impulsive cylindroid, and the corresponding instantaneous cylindroid, be known, we can then construct two systems of screws of the fifth order, such that an impulsive wrench on a given screw of one system will produce an instantaneous twist velocity about a determined screw on the other system.

It is interesting to note, in what way our knowledge of but *two*

corresponding pairs of impulsive screws and instantaneous screws just fails to give complete information with respect to every other pair. If we take any ray in space, and assign to it an arbitrary pitch, the screw so formed may be regarded as an impulsive screw, and the corresponding instantaneous screw will not, in general, be defined. There is, however, a particular pitch for each such screw, which will constitute it a member of the system of the fifth order. It follows that any ray in space, when it receives the proper pitch, will be such that an impulsive wrench, thereon delivered, would set any one of the singly infinite system of rigid bodies twisting about the same screw α.

IV.—The Geometrical Theory of three pairs of Screws.

It will be desirable, to complete this part of the investigation by showing how, when three pairs of corresponding impulsive screws and instantaneous screws are given, the instantaneous screw, corresponding to any impulsive screw, is geometrically constructed.

The solution depends upon the following proposition, which I have set down in its general form, though the application to be made of it is somewhat specialized.

Given any two independent systems of screws of the third order, P and Q. Let ω be any screw which does not belong either to P or to Q, then it is possible to find in one way, but only in one, a screw θ, belonging to P, and a screw ϕ, belonging to Q, such that ω, θ and ϕ shall all lie on the same cylindroid. This is proved as follows.

Draw the system of screws of the third order, P', which is reciprocal to P, and the system Q', which is reciprocal to Q. The screws belonging to P', which are at the same time reciprocal to ω, constitute a group reciprocal to four given screws. They, therefore, lie on a cylindroid which we call P_0. In like manner, since Q is a system of the third order, the screws that can be selected from it, so as to be at the same time reciprocal to ω, will also form a cylindroid which we call Q_0.

It is generally a possible and determinate problem to find, among the screws of a system of the third order, one screw which shall be reciprocal

to every screw, on an arbitrary cylindroid. For, take three screws from the system reciprocal to the given system of the third order, and two screws on the given cylindroid. As a single screw can be found reciprocal to any five screws, the screw reciprocal to the five just mentioned will be the screw now desired.

We apply this principle to obtain the screw θ, in the system P, which is reciprocal to the cylindroid Q_0. In like manner, we find the screw ϕ, in the system Q, which is reciprocal to the cylindroid P_0.

From the construction it is evident that the three screws θ, ϕ, and ω are all reciprocal to the two cylindroids P_0 and Q_0. This is, of course, equivalent to the statement that θ, ϕ, ω are all reciprocal to the screws of a system of the fourth order.

It follows that, θ, ϕ, ω must lie upon the same cylindroid. Thus, θ, ϕ are the two screws required, and the problem has been solved. It is easily seen, that there is only one such screw θ, and one such screw ϕ.

Or we might have proceeded as follows:—Take any three screws on P, and any three screws on Q. Then by a fundamental principle a wrench on ω can be decomposed into six component wrenches on these six screws. But the three component wrenches on P will compound into a single wrench on some screw θ belonging to P. The three component wrenches on Q will compound into a single wrench on some screw ϕ belonging to Q. Thus the original wrench on ω may be completely replaced by single wrenches on θ und ϕ. But this proves that θ, ϕ, and ω are co-cylindroidal.

In the special case of this theorem which we are now to use, one of the systems of the third order assumes an extreme type. It consists simply of all possible screws of infinite pitch. The theorem just proved asserts that in this case a twist velocity about any screw ω can always be replaced by a twist velocity about some one screw belonging to any given system of the third order P, together with a suitable velocity of translation.

In the problem before us we know three corresponding pairs of impulsive screws and instantaneous screws (η, α), (ξ, β), (ζ, γ), and we seek the impulsive screw corresponding to some fourth instantaneous screw δ.

It should be noticed that the data are sufficient but not redundant. We have seen how a knowledge of two pairs of corresponding impulsive screws

and instantaneous screws provided eight of the coordinate of the rigid body. The additional pair of corresponding screws only bring one further coordinate. For, though the knowledge of γ appropriate to a given ζ might seem five data, yet it must be remembered that the two pairs (η, α) and (ζ, γ) must fulfil the two fundamental geometrical conditions, and so must also the two pairs (ξ, β) and (ζ, γ); thus, as γ has to comply with four conditions, it really only conveys a single additional coordinate, which, added to the eight previously given, make the nine which are required for the rigid body. We should therefore expect that the knowledge of three corresponding pairs must suffice for the determination of every other pair.

Let the unit twist velocity about δ be resolved by the principles explained in this section into a twist velocity on some screw δ_0 belonging to α, β, γ, and into a velocity of translation on a screw δ_1 of infinite pitch.

We have already seen that the impulsive screw corresponding to δ_0 must lie on the system of the third order defined by η, ξ, and ζ, and that it is definitely determined. Let us denote by ψ this known impulsive screw which would make the body commence to twist about δ_0.

Let the centre of gravity be constructed as in the last section; then an impulsive force through the centre of gravity will produce the velocity of translation on δ_1. Let us denote by χ the screw of zero pitch on which this force lies.

We thus have χ as the impulsive screw corresponding to the instantaneous screw δ_1, while ψ is the impulsive screw corresponding to the instantaneous screw δ_0.

Draw now the cylindroids (χ, ψ) and (δ_1, δ_0). The first of these is the locus of the impulsive screws corresponding to the instantaneous screws on the second. As already explained, we can completely correlate the screws on two such cylindroids. We can, therefore, construct the impulsive screw on (χ, ψ) which corresponds to any instantaneous screw on (δ_1, δ_0). It is, however, obvious, from the construction, that the original screw δ lies on the cylindroid (δ_1, δ_0). Hence we obtain the impulsive screw which corresponds to δ as the instantaneous screw, and the problem has been solved.

We might, however, have proceeded as follows:—From the three given pairs of impulsive screws and instantaneous screws $\eta\alpha$, $\xi\beta$, $\zeta\gamma$ we can find other pairs in various ways. For example, draw the cylindroids (α, β) and (ξ, ζ); then select, by principles already explained, a screw δ on the first cylindroid, and its correspondent θ on the second. In like manner, from the cylindroids $(\alpha\gamma)$ and $(\eta\zeta)$, we can obtain another pair (ϕ, ϵ). We have thus five pairs of correspondents, $\eta\alpha$, $\xi\beta$, $\zeta\gamma$, $\theta\delta$, $\phi\epsilon$. Each of these will give a diameter of the momental ellipsoid, and the radius of gyration about that diameter. Thus we know the centre of the momental ellipsoid and five points on its surface. The ellipsoid can be drawn accordingly. Its three principal axes give the principal screws of inertia. All other pairs of correspondents can then be determined by the construction that will be given in Section VI.

V.—Identical Impulsive and Instantaneous Systems.

Suppose that a cylindroid be drawn through any two (not lying along the same principal axis) of the six principal screws of inertia of a free rigid body. If the body while at rest be struck by an impulsive wrench about any one of the screws of the cylindroid it will commence to move by twisting about a screw which also lies on the cylindroid. For, the given impulsive wrench can be replaced by two component wrenches on any two screws of the cylindroid. We shall, accordingly, take the component wrenches of the given impulse on the two principal screws of inertia which, by hypothesis, are contained on the cylindroid. Each of those components will, by the property of a principal screw of inertia, produce an instantaneous twist velocity about the same screw. But, the two twist velocities so generated can, of course, only compound into a single twist velocity on some other screw of the cylindroid. We have now to obtain the geometrical relations characteristic of the pairs of impulsive and instantaneous screws on such a cylindroid.

In previous memoirs, full discussions have been given of the relations between impulsive screws and instantaneous screws, when the movements of the body are confined, by geometrical constraint, to twists about the

screws on a cylindroid. The problem now before us is a special case, for though the movements are no other than twists about the screws on a cylindroid, yet this restriction, in the present case, is not the result of constraint. It arises from the fact that two of the six principal screws of inertia of the rigid body lie on the cylindroid, while the impulsive wrench is, by hypothesis, limited to the same surface.

To study the question, we shall make use of the representation employed in the Sixth Memoir. We have there shown how, when the several screws on the cylindroid are represented by points on the circumference of a circle, various dynamical problems can be solved with simplicity and convenience. For example, when the impulsive screw is represented by one point on the circle, and the instantaneous screw by another, we see how these points are connected by geometrical construction.

In the case of the unconstrained body, which is that now before us, it can be shown that, whenever the pitch of an instantaneous screw is zero, the corresponding impulsive screw must be at right angles thereto.

In the circular representation, the angle between any two screws is equal to the angle subtended in the representative circle by the chord whose extremities are the representatives of the two screws. Two screws, at right angles, are consequently represented by the extremities of a diameter of the representative circle. If, therefore, we take A, B, two points on the circle, to represent the two screws of zero pitch, then the two points, P and Q, diametrically opposite to them, are the points indicating the corresponding impulsive screws.

The two principal screws of the cylindroid X and Y are, in this case, the principal screws of inertia. Each of them, when regarded as an impulsive screw, coincides with its corresponding instantaneous screw. The diameter XY bisects the angle between AP and BQ.

It is shown (Sixth Memoir) that the points which represent the instantaneous screws, and the points which represent the corresponding impulsive screws, form two homographic systems. A well known geometrical principle asserts, that if each point on a circle be joined to its homographic correspondent, the chord will envelop a conic which has double contact with the circle. It is easily seen that, in the present

case, the conic must be the hyperbola which touches the circle at the ends of the diameter XY, and has the rays AP and BQ for its asymptotes. The hyperbola is completely defined by these conditions, so that the pairs of correspondents are uniquely determined.

Every tangent, IST, to this hyperbola will cut the circle in two points, I and S, such that S is the point corresponding to the impulsive screw, and I is the point which marks out the instantaneous screw. We thus obtain a concise geometrical theory of the connexion between the pairs of corresponding impulsive screws and instantaneous screws on a cylindroid which contains two principal screws of inertia.

For completeness, it may be necessary to solve the same problem when the cylindroid is defined by two principal screws of inertia lying along the same principal axis of the rigid body. It is easy to see that if, on the principal axis, whose radius of gyration was a, there lay any instantaneous screw whose pitch was p_a, then the corresponding impulsive screw would be also on the same axis, and its pitch would be p_η where $p_a \times p_\eta = a^2$.

Let us now take the case of the system of screws of the third order, which contains three of the six principal screws of inertia of a free rigid body.

Any impulsive wrench, which acts on a screw of a system of the third order, can be decomposed into wrenches on any three screws of that system, and consequently, in the present case, on the three principal screws of inertia, which it has been made to contain. Each of these component wrenches will, from the property of a principal screw of inertia, generate an initial twist velocity of motion around the same screw. The three twist velocities, thence arising, can be compounded into a single twist velocity about some other screw of the system. We desire to obtain a distinct geometrical conception of the relation between each such resulting instantaneous screw and the initial impulsive screw.

It has been explained in the Eighth Memoir how the several points in a plane are in correspondence with the several screws which constitute a system of the third order. It was further shown, that if by the imposition of geometrical constraints, the freedom of a rigid body was

limited to twisting about the several screws of the system of the third order, a geometrical construction could be obtained for determining the point corresponding to any instantaneous screw, when the point corresponding to the appropriate impulsive screw was known.

We have now to introduce the simplification of the problem, which results when three of the principal screws of inertia of the body belong to the system. But a word of caution, against a possible misunderstanding, is first necessary. It is one of the most important principles in the Theory of Screws, that when a rigid system has freedom of the n^{th} order, there will always be, in the system of screws expressing that freedom, n screws such that an impulsive wrench administered on any one of those screws, will immediately make the body begin to move by twisting about the same screw. We have called these the *n principal screws of inertia.* They have occupied much attention in the preceding Memoirs, and they will again claim our attention in the present.

But in the case immediately under consideration, the rigid body was supposed free, and it had, therefore, six principal screws of inertia. The system of the third order, at present before us, is one which contains three of these principal screws of inertia of the free body. Such a system of screws possesses the property, that an impulsive wrench on any screw belonging to it will set the body twisting about a screw which also belongs to the same system. This is the case even though, in the total absence of constraints, there is no kinematical difficulty about the body twisting about any screw whatever.

As there are no constraints, we know that each instantaneous screw, of zero pitch, must be at right angles to the corresponding impulsive screw. This condition will enable us to adjust the particular homography in the plane wherein each pair of correspondents represents an impulsive screw and the appropriate instantaneous screw.

The conic, which is the locus of points corresponding to the screws of a given pitch p, has as its equation (see Eighth Memoir)

$$p_1\theta_1^2 + p_2\theta_2^2 + p_3\theta_3^2 - p(\theta_1^2 + \theta_2^2 + \theta_3^2) = 0.$$

All the conics corresponding to the various values of p have a common

conjugate triangle. The vertices of that triangle correspond to the three principal screws of inertia.

The three points just found are the double points of the homography which connects the points representing the impulsive screws with those representing the instantaneous screws. Let us take the two conics of the system, corresponding to $p = 0$ and $p = \infty$. They are

$$p_1\theta_1^2 + p_2\theta_2^2 + p_3\theta_3^2 = 0, \qquad \text{(i)}$$

$$\theta_1^2 + \theta_2^2 + \theta_3^2 = 0. \qquad \text{(ii)}$$

Two conjugate points to conic (i) denote two reciprocal screws. Two conjugate points to conic (ii) denote two screws at right angles.

Let A be any point representing an instantaneous screw. Take the polar of A, with respect to conic (i). Let P be the pole of this ray, with respect to conic (ii).

Then P will correspond to the impulsive screw, while A corresponds to the appropriate instantaneous screw. For this is clearly a homography of which A and P are two correspondents. Further, the double points of this homography are the vertices of the common conjugate triangle to conics (i) and (ii). If A lie on (i), then its polar is the tangent to (i); and as every point on this polar will be conjugate to P, with respect to conic (ii), it follows that A and P are conjugate, with respect to (ii)—that is, A and P are correspondents of a pair of screws at right angles. As the pitch of the screw, corresponding to A, is zero, we have thus obtained the solution of our problem.

VI.—Fundamental Problem with Free Body.

I now give the geometrical solution of the fundamental problem in Dynamics which may be thus stated—

A perfectly free body at rest is struck by an impulsive wrench upon a given screw. It is required to determine the instantaneous screw about which the body will commence to twist.

It was shown in the Third Memoir that a certain solution of the problem of finding the instantaneous screw when the impulsive screw was given

could be thus obtained. Decompose the given impulsive wrench supposed of unit magnitude on the screw η into six components $\eta_1 \ldots \eta_6$ on the six principal screws of inertia. These six quantities are the coordinates of η. If $p_1, p_2, \ldots p_6$ be the pitches of the six principal screws of inertia, then the components of the resulting twist velocity on the same screws are proportional to $\frac{1}{p_1}\eta_1, \frac{1}{p_2}\eta_2, \ldots \frac{1}{p_6}\eta_6$. It was only necessary to compound these six twist velocities and thus obtain the resulting screw α. Thus the connexion between the impulsive screw and the instantaneous screw was determined with all needful convenience for the purpose of calculation.

But from the time when my attention was first given to the Theory of Screws I have sought for a geometrical construction by which, when the impulsive screw was known, the instantaneous screw could be determined, or *vice versâ*. I have at last obtained such a construction, and here proceed to set it down.

The rigid body being given, its three principal axes are to be drawn through its centre of gravity. The radii of gyration a, b, c about these axes are to be found. On the first principal axis two screws of pitches $+a$ and $-a$ respectively are to be placed. Similarly screws of pitches $+b$, $-b$, and $+c$, $-c$ are to be placed on the other two principal axes. These are, of course, the six principal screws of inertia: call them $A_0, A_1, A_2, A_3, A_4, A_5$. We then draw the five cylindroids

$$A_0A_1,\ A_0A_2,\ A_0A_3,\ A_0A_4,\ A_0A_5.$$

It is always possible to find one screw on a cylindroid reciprocal to any given screw. In certain cases, however, of a special nature, more than a single screw can be so found. Under such circumstances the present process is inapplicable, but the exceptional instances will be dealt with presently.

Choose on the cylindroid A_0A_1 a screw θ_1 which is reciprocal to the given impulsive screw η which is, of course, supposed to lie anywhere and be of any pitch.

In like manner, choose on the other four cylindroids screws θ_2, θ_3, θ_4, θ_5, respectively, all of which are also reciprocal to η.

Let us now think of θ_1 as an instantaneous screw; it lies on the cylindroid A_0A_1, and this cylindroid contains two principal screws of inertia. It follows from Section V. that the corresponding impulsive screw ϕ_1 lies on the same cylindroid. That screw ϕ_1 can be determined by the construction there given. In like manner we construct on the other four cylindroids the screws ϕ_2, ϕ_3, ϕ_4, ϕ_5, which are the impulsive screws corresponding respectively to θ_2, θ_3, θ_4, θ_5, as instantaneous screws.

Consider then the two pairs of corresponding impulsive screws and instantaneous screws (η, α) and (ϕ_1, θ_1). We have arranged the construction so that θ_1 is reciprocal to η. Hence, by the fundamental principle so often employed, α and θ_1 are conjugate screws of inertia, so that α must be reciprocal to ϕ_1.

In like manner it can be proved that the instantaneous screw α for which we are in search must be reciprocal to ϕ_2, ϕ_3, ϕ_4, ϕ_5. We have thus discovered five screws ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, to each of which the required screw α must be reciprocal. But it is a fundamental point in the theory that the single screw reciprocal to five screws can be constructed geometrically (First Memoir). Hence α is determined, and the geometrical solution of the problem is complete.

It remains to examine the failure in this construction which arises when any one or more of the five screws $\phi_1 \ldots \phi_5$ becomes indeterminate. This happens when η is reciprocal to *two* screws on the cylindroid in question. In this case η is reciprocal to every screw on the cylindroid. Any one of such screws might be taken as the corresponding ϕ, and, of course, θ would have been also indefinite, and α could not have been found. In this case η would have been reciprocal to the two principal screws of inertia, suppose A_0, A_1 which the cylindroid contained. Of course still more indeterminateness would arise if η had been also reciprocal to other screws of the series A_0, A_1, A_2, A_3, A_4, A_5. No screw could, however, be reciprocal to all of them. If η had been reciprocal to five, namely, A_1, A_2, A_3, A_4, A_5, then η could be no screw other than A_0, because the six principal screws of inertia are coreciprocal; η would then be its own instantaneous screw, and the problem would be solved.

We may therefore, under the most unfavourable conditions, take η to

be reciprocal to four of the principal screws of inertia A_0, A_1, A_2, A_3, but not to A_4 or A_5. We now draw the five cylindroids, A_0A_4, A_1A_4, A_2A_4, A_3A_4, A_0A_5. We know that η is reciprocal to no more than a single screw on each cylindroid. We therefore proceed to the construction as before, first finding $\theta_1 \ldots \theta_5$, one on each cylindroid; then deducing $\phi_1 \ldots \phi_5$, and thus ultimately obtaining α.

Thus the general problem has been solved. We have only to remark that when once the screw α has been determined the twist velocity with which the body commences to move in consequence of an impulsive wrench on η may be thus calculated.

Let η'' be the intensity of the wrench on η expressed in absolute units.

Let e be the brief time during which this wrench acts.

Let M be the mass of the body.

Then the acquired twist velocity $\dot{\alpha}$ is thus expressed

$$\dot{\alpha} = \frac{e}{M}\eta'' \frac{\cos \hat{\alpha\eta}}{p_\alpha}.$$

VII.—FUNDAMENTAL PROBLEM WITH CONSTRAINED BODY.

If the rigid body have only a single degree of freedom, then the only movements of which it is capable are those of twisting to and fro on a single screw α. If the impulsive wrench η which acted upon the body happened to be reciprocal to α, then no movement would result. The forces would be destroyed by the reactions of the constraints. In general, of course, the impulsive screw η will not be reciprocal to α. A twisting motion about α will therefore be the result. All that can be said of the instantaneous screw is that it can be no possible screw but α.

In the next case the body has two degrees of freedom which, as usual, we consider to be of the most general type. It is required to obtain a construction for the instantaneous screw α about which a body will commence to twist in consequence of an impulsive wrench η.

The peculiarity of the problem when the notion of constraint is introduced depends on the circumstance that, though the impulsive screw may

be situated anywhere and be of any pitch, yet that as the body is restrained to only two degrees of freedom, it can only move by twisting about one of the screws on a certain cylindroid. We are, therefore, to search for the instantaneous screw on the cylindroid expressing the freedom.

Let A be the given cylindroid. Let B be the system of screws of the fourth order reciprocal to that cylindroid. If the body had been free it would have been possible to determine, in the manner explained in the last section, the impulsive screw corresponding to each screw on the cylindroid A. Let us suppose that these impulsive screws are constructed. They will all lie on a cylindroid which we denote as P. In fact, if any two of such screws had been found, P would of course have been defined by drawing the cylindroid through those two screws.

Let Q be the system of screws of the fourth order which is reciprocal to P. Select from Q the system of the third order Q_1 which is reciprocal to η. We can then find one screw η_1 which is reciprocal to the system of the fifth order formed from A and Q_1. It is plain that η_1 must belong to B, as this contains every screw reciprocal to A.

Take also the one screw on the cylindroid A which is reciprocal to η, and find the one screw η_2 on the cylindroid P which is reciprocal to this screw on A.

Since η, η_1, and η_2 are all three reciprocal to the system of the fourth order formed by A_1 and Q_1, it follows that η, η_1, and η_2 must all lie on the same cylindroid. We can therefore resolve the original wrench on η into two component wrenches on η_1 and η_2.

But it is of the essence of the theory (First Memoir) that the reactions of the constraints by which the motion of the body is limited to twists about screws on the cylindroid A must be wrenches on the reciprocal system B. So far, therefore, as the body thus constrained is concerned, the reactions of the constraints will neutralize the wrench on η_1. Thus the wrench on η_2 is the only part of the impulsive wrench which need be considered.

But we already know from the construction that an impulsive wrench on η_2 will produce an instantaneous twist velocity about a known screw α on A. Thus we have found the initial movement, and the investigation is geometrically complete.

We next consider the case in which a rigid body has three degrees of freedom. We require, as before, to find a geometrical construction for the instantaneous screw α corresponding to a given impulsive screw η.

Let A be the system of screws of the third order about which the body is free to twist. Let B be the system of screws of the third order reciprocal to A. We must first construct the system of the third order P which consists of the impulsive screws that would have made the body, if perfectly free, twist about the several screws of A.

As already explained we can, in one way, but only in one way, resolve the original wrench on η into wrenches η_1 on B, and η_2 on P. The former is destroyed by the reactions of the constants. The latter gives rise to a twist velocity about a determinate screw on A. Thus the problem has been solved.

We can obviously extend a similar line of reasoning to the cases where the body had freedom of the remaining degrees. It will, however, be as simple to write the general case at once.

Let A be a system of screws of the n^{th} order, about which a body is free to twist, any other movements being prevented by constraints. If the body receive an impulsive wrench, on any screw η, it is required to determine the instantaneous screw, of course belonging to A, about which the body will commence to twist.

Let B be the system of screws of the $(6-n)^{th}$ order, reciprocal to A. The wrenches arising from the reaction of the constraint must, of course, be on the screws of B.

Let P be the system of screws of the n^{th} order, which, in case the body had been free, would have been the impulsive screws, corresponding to the instantaneous screws on A.

Let Q be the system of screw of the $(6-n)^{th}$ order, which are reciprocal to P.

Take from A the system of the $(n-1)^{th}$ order, reciprocal to η, and call it A_1.

Take from Q the system of the $(5-n)^{th}$ order, reciprocal to η, and call it Q_1.

As A is of the n^{th} order, and Q_1 of the $(5-n)^{th}$, they together define a

system of the fifth order. Let η_1 be the single screw, reciprocal to this system, of the fifth order.

As A_1 is of the $(n-1)^{th}$ order, and Q is of the $(6-n)^{th}$ order, they together define a system of the fifth order. Let η_2 be the single screw, reciprocal to this system, of the fifth order.

η_1 is reciprocal to A_1 of the $(n-1)^{th}$ order, because A_1 forms part of A. It is reciprocal to Q_1 of the $(5-n)^{th}$ order, because it was so made by construction. Thus η_1 is reciprocal to both A_1 and Q_1, that is to a system of the fourth order.

In like manner, it can also be shown that η_2 is reciprocal to both A_1 and Q_1.

A_1 and Q_1 were originally chosen so as to be reciprocal to η. It thus appears that the three screws, η, η_1, η_2, are all reciprocal to the same system of the fourth order. They are, therefore, co-cylindroidal.

The initial wrench on η can therefore be fully replaced by its two components on η_1 and η_2. The former of these is destroyed by the reaction of the constraints. The latter gives rise to an initial movement on a determined screw of A. Thus, the most general problem of the effect of an impulsive wrench on a constrained rigid body, has been solved geometrically.

The special case, where the rigid body has freedom of the fifth order, may be viewed as follows.

Let ρ be the screw reciprocal to the screw system of the fifth order, about which the body is free to twist.

Let λ be the instantaneous screw, determined in the way already explained, about which the body, had it been free, would have commenced to twist in consequence of an impulsive wrench on ρ.

Let η be any screw on which an impulsive wrench is imparted, and let α be the corresponding instantaneous screw, about which the body would have begun to twist had it been free.

Draw the cylindroid through α and λ, and choose on this cylindroid the screw μ, which is reciprocal to ρ.

Then μ is the instantaneous screw about which the body commences to twist, in consequence of the impulsive wrench on η.

For (η, ρ) is a cylindroid of impulsive screws, and (α, λ) are the corresponding instantaneous screws. As μ is reciprocal to ρ, it belongs to the system of the fifth order. The corresponding impulsive screw must lie on (η, ρ). The actual instantaneous motion could therefore have been produced by impulsive wrenches on η and ρ. The latter would, however, be neutralized by the reactions of the constraints. We therefore find that η is the impulsive screw, corresponding to α as the instantaneous screw.

It has been shown in the Fourth Memoir how, by the aid of the geometrical entity termed a screw-chain, the Theory of Screws can be extended from one rigid body to any system of such bodies. By the application of the results of the present section we obtain a geometrical solution of the instantaneous motion of any material system whatever, however connected or however constrained under the action of a system of forces of the most general type.

VIII.—PRINCIPAL SCREWS OF INERTIA OF CONSTRAINED BODY.

There is no more important theorem in this part of the Theory of Screws than that which affirms that for a rigid body, with n degrees of freedom, there are n screws, such that if the body when quiescent receives an impulsive wrench about one of such screws, it will immediately commence to move by twisting about the same screw.

We shall show how the principles, already explained, will enable us to construct these screws.

We commence with the case in which the body has two degrees of freedom. We take three screws, η, ξ, ζ, arbitrarily selected on the cylindroid, which expresses the freedom of the body. We can then determine, by the preceding investigation, the three instantaneous screws, α, β, γ, on the same cylindroid, which correspond, respectively, to the impulsive screws. Of course, if η happened to coincide with α, or ξ with β, or ζ with γ, one of the principal screws of inertia would have been found. But, in general, such pairs will not coincide. We have to show how, from the knowledge of these such pairs, in general, the two principal screws of inertia can be found.

The simplest method is to employ the circular representation of the points on the cylindroid, as explained in the Sixth Memoir. The impulsive screws are represented by one system of points, the corresponding instantaneous screws are represented by another system of points. It is an essential principle, that the two systems of points, so related, are homographic. The discovery of the principal screws of inertia is thus reduced to the well-known problem of the discovery of the double points of two homographic systems on a circle.

The simplest method of solving this problem is to regard the six points, suitably arranged, as the vertices of a hexagon; then the Pascal line of the hexagon intersects the circle in two points which are the points corresponding to the principal screws of inertia.

We next investigate the principal screws of inertia of a body which has three degrees of freedom. We have first, by the principles already explained, to discover four pairs of correspondents. When four such pairs are known, the pricipal screws of inertia can be constructed. Perhaps the best method of doing so is to utilize the plane correspondence, as explained in the Eighth Memoir. The corresponding systems of impulsive screws and instantaneous screws, in the system of the third order, are then represented by the homographic systems of points in the plane. When four pairs of such correspondents are known, we can construct as many additional pairs as may be desired.

Let α, β, γ, δ be four points in the plane, and let η, ξ, ζ, θ be the points corresponding, so that η represents the impulsive screw, and α the instantaneous screw, and similarly for the other pairs. Let it be required to find the impulsive screw ϕ, which corresponds to any fifth instantaneous screw ϵ. Since anharmonic ratios are the same in two corresponding figures, we have

$$\alpha(\beta, \gamma, \delta, \epsilon) = \eta(\xi, \zeta, \theta, \phi),$$

thus we get one ray $\eta\phi$, which contains ϕ. We have also

$$\beta(\alpha, \gamma, \delta, \epsilon) = \xi(\eta, \zeta, \theta, \phi),$$

which gives a second ray $\xi\phi$, containing ϕ, and thus ϕ is known.

The points on a conic, regarded as belonging to one system, will correspond to the points on another conic, regarded as belonging to the other system. Let us then take five points of one system, and the five corresponding points belonging to the other system. Each set of five points will determine one of two corresponding conics. The common conjugate triangle of the two conics will have, as its vertices, the double points of the system. Thus we obtain the three principal screws of inertia in the case of the system of the third order.

If a rigid body be free to move about screws of any system of the fourth order, we may determine its four principal screws of inertia as follows.

We correlate the several screws of the system of the fourth order with the points in space. The points representing impulsive screws will be a system homographic with the points representing the corresponding instantaneous screws. If we have five pairs of correspondents (α, η), (β, ξ), (γ, ζ), (δ, θ), (ϵ, ϕ), in such a homography we can at once determine the correspondent ψ to any other screw λ.

Draw a pencil of four planes through α, β, and the four points γ, δ, ϵ, λ, respectively.

Draw also a pencil of four planes through η, ξ, and the four points ζ, θ, ϕ, ψ, respectively. These two pencils will be equianharmonic. Thus we discover one plane which contains ψ. In like manner, we can draw a pencil of planes through α, γ, and β, δ, ϵ, λ, respectively, and the equianharmonic pencil through η, ζ, and ξ, θ, ϕ, ψ, respectively. Thus we obtain a second plane which passes through ψ. A third plane may be found by drawing the pencil of four planes through α, δ, and the four points β, γ, ϵ, λ, respectively, and then constructing the equianharmonic pencil through η, θ, and ξ, ζ, ϕ, ψ, respectively. From the intersection of these three planes, ψ is known.

In the case of homographic systems in space, there are, of course, four double points. These may be determined as follows. We can take any nine points in one system, and their correspondents. Each set of nine points will determine the quadric which passes through them. Thus we obtain two corresponding quadrics. The vertices of the common conjugate tetrahedron of the two quadrics are the double points of the homography.

These correspond to the principal screws of inertia, which are accordingly determined.

In the case of freedom of the fifth order, the geometrical analogies which have hitherto sufficed are not available. We have to fall back on the general fact that the impulsive screws and the corresponding instantaneous screws form two homographic systems. There are five double screws belonging to this homography. These are the principal screws of inertia.

From the theory of screw-chains as developed in the Fourth Memoir, it was shown that any quiescent mechanical system whatever, with n degrees of freedom, has n principal screw-chains of inertia such that an impulsive wrench delivered on any one of those screw-chains caused the system to begin to twist about the same screw-chain. The principles explained in the present section show how these n screw-chains of inertia can be geometrically constructed.

IX.—System of the Third Order.

It being given that a certain system of screws of the third order, P, is the locus of impulsive screws corresponding to another given system of the third order, A, as instantaneous screws, it is required to correlate the corresponding pairs on the two systems.

We have already had frequent occasion to use the result demonstrated in the last Memoir, namely, that when two impulsive and instantaneous cylindroids were known, we could immediately arrange the several screws in their corresponding pairs. We have now to demonstrate that the same can be done when we are given an impulsive system of the third order, and the corresponding instantaneous system.

It has first to be shown, that the proposed problem is a definite one. The data before us are sufficient to discriminate the several pairs of screws, that is to say, the data are sufficient to point out in one system the correspondent to any specified screw in the other system. We have also to show that there is no ambiguity in the solution. It is not possible that there could be more than one arrangement of corresponding pairs.

Let α, β, γ be three instantaneous screws from A, and let their corresponding impulsive screws be η, ξ, ζ, in P. In the choice of a screw from a system of the third order, there are two disposable quantities, so that, in the selection of three correspondents in P, to three given screws in A, there would be, in general, six disposable coordinates. But the fact that α, η and β, ξ are two pairs of correspondents necessitates, as we know, the fulfilment of two identical conditions among their coordinates. As there are three pairs of correspondents, we see at once that there are six equations to be fulfilled. These are exactly the number required for the determination of η, ξ, ζ, in the system P.

To the same conclusion, we might have been conducted by a different line of reasoning. It is known that, for the complete specification of a system of the third order, nine coordinates are necessary. This is exactly the same number as is required for the specification of a rigid body. If, therefore, we are given that P, a system of the third order, is the collection of impulsive screws, corresponding to the instantaneous screws in the system A, we are given nine data towards the determination of a rigid body, for which A and P would possess the desired relation. It therefore follows that we have nine equations, while the rigid body involves nine unknowns. Thus we are led to expect that the number of bodies, for which the arrangement would be possible, is finite. When such a body was determined, then of course the correlation of the screws on the two systems is immediately accomplished. Here then, again, we see that the general problem of correlating the screws on any two given systems of the third order, A and P, into possible pairs of impulsive screws and instantaneous screws, does not admit of more than a finite number of solutions.

We are now to prove that this finite number of solutions cannot be different from unity.

For, let us suppose that a screw λ, belonging to A, had two screws θ and ϕ, as possible correspondents in P. This could, of course, in no case be possible with the same rigid body. We shall show that it could not even be possible with two rigid bodies, M_1 and M_2. For, if two bodies could do what is suggested, then it can be shown that there are a

singly infinite number of possible bodies, each of which would afford a different solution of the problem.

We could design a body in the following manner:—

Increase the density of every element of M_1 in the ratio of $\rho_1 : 1$, and call the new mass M'_1.

Increase the density of every element of M_2 in the ratio of $\rho_2 : 1$, and call the new mass M'_2.

Let the two bodies, so altered, be conceived bound rigidly together by bonds which are regarded as imponderable.

Let ψ be any screw lying on the cylindroid (θ, ϕ). Then the impulsive wrench of intensity, ψ'' on ψ, may be decomposed into components

$$\psi'' \frac{\sin(\psi - \phi)}{\sin(\theta - \phi)} \text{ on } \theta, \text{ and } \psi'' \frac{\sin(\theta - \psi)}{\sin(\theta - \phi)} \text{ on } \phi.$$

If the former had been applied to M'_1 for the short time ϵ, it would have generated a twist velocity about α represented by

$$\psi'' \frac{\sin(\psi - \phi)}{\sin(\theta - \phi)} \frac{\epsilon}{M'_1} \frac{\cos \hat{\theta\alpha}}{p_\alpha}.$$

If the latter had been applied, for the small time ϵ, to the body M'_2, it would have generated a twist velocity about the same screw, α, equal to

$$\psi'' \frac{\sin(\theta - \psi)}{\sin(\theta - \phi)} \frac{\epsilon}{M'_2} \frac{\cos \hat{\phi\alpha}}{p_\alpha}.$$

Suppose that these two twist velocities are equal, it is plain that the original wrench on ψ would, if it had been applied to the composite rigid body, produce a twisting motion about α. The condition is

$$\frac{\sin(\psi - \phi)}{M'_1 \cos \hat{\phi\alpha}} = \frac{\sin(\theta - \psi)}{M'_2 \cos \hat{\theta\alpha}}.$$

We thus obtain $\tan \psi$ in terms of $M'_1 : M'_2$. As the structure of the composite body changes by alterations of the relative values of ρ_1 and ρ_2, so will ψ move over the various screws of the cylindroid (θ, ϕ).

This result shows that, if three screws, α, β, γ, be given, then the possible impulsive screws, η, ξ, ζ, which shall correspond there to in a given system of the third order P, are uniquely determined.

For, suppose that a second group of screws, η', ξ', ζ', could also be determined which fulfilled the same property. We have shown how another rigid body could be constructed so that another screw, ψ, could be found on the cylindroid (η, η'), such that an impulse thereon given would make the body twist about α. It is plain that, for this body also, the impulsive wrench, corresponding to β, would be some screw on the cylindroid (ξ, ξ'). But all screws on this cylindroid belong to the system P. In like manner, the instantaneous screw γ would correspond for the composite body to some screw on the cylindroid (ζ, ζ'). Hence it follows that, for each different value of the ratio $\rho_1 : \rho_2$, we would have a different set of impulsive screws for the instantaneous screws α, β, γ. We thus find that, if there were more than one set of such impulsive screws to be found in the system P, there would be an infinite number of such sets. But we have already shown that the number of sets must be finite. Hence there can only be one set of screws, η, ξ, ζ, in the system P, which could be impulsive screws corresponding to the instantaneous screws α, β, γ. We are thus led to the following theorem :—

Given any two systems of screws of the third order. It is always possible, in one way, but only in one, to design place and orientate a rigid body such that, if that body, while at rest and unconstrained, receive an impulsive wrench about any screw of the first system, the instantaneous movement will be a twist about a screw of the second system.

The two systems of corresponding impulsive and instantaneous screws on the two systems of the third order, form two homographic systems. There are, of course, infinite varieties in the possible homographic correspondences between the screws of two systems of the third order. The number of such correspondences is just so many as the number of possible homographic correspondences of points in two planes. There is, however, only one correspondence which will fulfil the peculiar requirements when one of the systems expresses the instantaneous screws, and the other the appropriate impulsive screws.

If we are given one pair of corresponding impulsive and instantaneous screws, the body is not by such data fully determined. We are only given five coordinates, and four more remain, therefore, unknown. If we are given two corresponding impulsive cylindroids and instantaneous cylindroids, the body is still not completely specified. We have seen how eight of its coordinates are determined, but there is still one remaining indeterminate. If we are given a system of the fourth order of impulsive screws, and the corresponding system of the fourth order of instantaneous screws, the body, as in the other cases remaining perfectly free, there are also, as we shall see in the next section, a singly infinite number of rigid bodies which fulfil the necessary conditions. In like manner, it will appear that, if we are given a system of the fifth order consisting of impulsive screws, and a corresponding system of the fifth order consisting of instantaneous screws, the body has really as much indeterminateness as if we had only been given a single pair of corresponding screws.

But the case of two systems of the third order is exceptional, in that when it is known that one of these is the locus of the instantaneous screws, which correspond to the screws of the other system regarded as impulsive screws, the rigid body for which this state of things is possible is completely and uniquely specified.

X.—Systems of the Fourth and Fifth Orders.

We here develop more fully the concluding part of the last section. An additional principle is involved, which may perhaps be best explained in answer to the following query :—

Given a system of the fourth order A and another system of the fourth order P. If it be known that the latter is the locus of the screws on which must lie the impulsive wrenches which would, if applied to a free rigid body, cause instantaneous twist velocities about the several screws on A, what can be inferred as to the rigid body ?

Let A' be the cylindroid which is composed of the screws reciprocal to A. Let P' be the cylindroid which is composed of the screws reciprocal

to P. Let P_1, P_2, P_3, P_4 be any four impulsive screws on P. Let A_1, A_2, A_3, A_4 be the four corresponding instantaneous screws on A.

Take any screw α on the cylindroid P'. Let η be the corresponding impulsive screw. Since α is reciprocal to all the screws on P it must be reciprocal to P_1. It follows from the fundamental property of conjugate screws of inertia, that η must be reciprocal to A_1. In like manner we can show that η is reciprocal to A_2, A_3, and A_4. It follows that η is reciprocal to the whole system A, and therefore must be contained in the reciprocal cylindroid A'. Hence we obtain the following remarkable result which is obviously generally true, though our proof has been stated for the system of the fourth order only.

Let P and A be any two systems of screws of the n^{th} order, and P' and A' their respective reciprocal systems of the $(6-n)^{th}$ order; then, if P be the collection of impulsive screws corresponding severally to the screws of A as the instantaneous screws for a certain free rigid body; then, for the same free rigid body A' will be the locus of the impulsive screws which correspond to the screws of P' as instantaneous screws.

Thus we see that when we are given two systems of the fourth order P and A as correspondingly impulsive and instantaneous, we can immediately infer that, for the same rigid body, the screws on the cylindroid A' are the impulsive screws corresponding to the instantaneous screws on the cylindroid P'.

We can now make use of that instructive theorem which declares that when two given cylindroids are known to stand to each other in this peculiar relation, we are then able, without any further information, to mark out on the cylindroids the corresponding pairs of screws. We can then, as shown in Section III., determine the centre of gravity of the rigid body on which the impulsive wrenches act. We can find a triad of conjugate diameters of the momental ellipsoid, and the radii of gyration about two of those diameters. Hence we have the following result:—

If it be given that a certain system of the fourth order is the locus of the impulsive screws corresponding to the instantaneous screws on another given system of the fourth order, the body being quite unconstrained, we can then determine the centre of the gravity of the body, we can draw a

triad of the conjugate diameters of its momental ellipsoid, and we can find the radii of gyration about two of those diameters.

There is still one undetermined element in the rigid body, namely, the radius of gyration about the remaining conjugate diameter. The data before us are not adequate to the removal of this indefiniteness. It must be remembered that what the data amount to in such a case are the data pertaining to the system A. The number of data necessary to define a system of the n^{th} order is $n(6-n)$. If, as in the present case $n = 4$, the number of data is 8. We are thus one short in the number of data necessary to specify a rigid body. Thus we confirm the result previously obtained. We can assert that for any one of the singly infinite number of rigid bodies which fulfil the necessary conditions, the system A will be the locus of the instantaneous screws which correspond to the screws of the system P as impulsive screws.

Though in the two cylindroids A' and P' we are able to establish the several pairs of correspondents quite definitely, yet we must not expect, with the data before us, to be able to correlate the pairs of screws in A and P definitely. If this could be done then the rigid body would be quite determinate, which we know is not the case. There is, however, only a single indeterminate element left in the correlation of the screws in A with the screws of P. This we prove as follows:—

Let ϕ be any screw of P on which an impulsive wrench is to act. Let δ be the instantaneous screw in A about which the movement commences. We shall now show that though δ cannot be completely defined, yet it can be shown to be limited to a certain cylindroid.

Let G be the centre of gravity. Then we know that an impulsive force directed through G will generate a movement of translation in a direction parallel to the force. Such a movement may, of course, be regarded as a twist about a screw of infinite pitch.

Draw through G a plane normal to ϕ. Any screws of infinite pitch in this plane will be reciprocal to ϕ. It follows from the laws of conjugate screws of inertia that the impulsive forces in this plane, by which translations could be produced, must lie on screws of zero pitch which are reciprocal to δ. Take any two of such screws: then we know that δ is reciprocal to

these two screws and also to P'. It follows that δ is reciprocal to the screws of a determinate system of the fourth order, and therefore δ must lie on a determined cylindroid.

We may commence to establish the correspondence between P and A by first choosing some arbitrary screw ϕ on P, and then drawing the cylindroid on A, on which we know that the instantaneous screw corresponding to P must lie. Any screw on this cylindroid may be selected as the instantaneous screw which corresponds to ϕ. Once that screw δ had been so chosen there can be no further ambiguity. The correspondent in A to every other screw in P is completely known. To show this it is only necessary to take two pairs from A' and P' and the pair just found. We have then three corresponding pairs. It has been shown in Section IV. how, in this case, the correspondence is completely determined.

Of course the fact that δ is limited to a cylindroid is connected with the fact that in this case the rigid body had one indeterminate element. For each of the possible rigid bodies δ would occupy a different position on its cylindroidal locus.

It remains to consider the case where two screw systems of the fifth order are given, it being known that one of them P is the locus of the impulsive screws which correspond to the several screws of the other system A regarded as instantaneous screws.

Let P' be the screw reciprocal to P, and A' the screw reciprocal to A. Then from the theorem of this section it follows that an impulsive wrench on A' would make the body commence to move by twisting about P'. We thus know five of the coordinates of the rigid body. There remain four indeterminate elements.

Hence we see that, when the only data are the two systems P and A, there is a fourfold infinity in the choice of the rigid body. There are consequently four arbitrary elements in designing the correspondence between the several pairs of screws in the two systems.

We may choose any two screws on P, η, ξ, and assume as their two correspondents in A any two arbitrary screws α and β, provided of course that the three pairs A', B', η, α, and ξ, β fulfil the six necessary geometrical conditions. Two of these conditions have been already fulfilled from

the fact that A' and P' are the reciprocals to the systems A and P. This leaves four conditions to be fulfilled in the choice of α and β. As each of these belongs to a system of the fifth order there will be four coordinates required for its complete specification. Therefore there will be eight disposable quantities in the choice of α and β. Four of these will be utilized in making them fulfil the geometrical conditions, so that four others may be arbitrarily selected. When these are chosen we have four coordinates of the rigid body which, added to the five data provided by A' and P', completely define the rigid body.

We may put the result of this discussion in the following manner:—

If we are given two systems of the first, or the second, or the third order of corresponding impulsive screws and instantaneous screws, all the corresponding pairs are determined. There is no arbitrary element in the correspondence. There is no possible rigid body which would give any different correspondence.

If we are given two systems of the fourth order of corresponding impulsive screws and instantaneous screws then subject to the observance of the essential geometrical conditions, we can select *one* pair of correspondents arbitrarily in the two systems, and find one rigid body to fulfil the requirements.

If we are given two systems of the fifth order of corresponding impulsive screws and instantaneous screws then subject to the observance of the geometrical conditions, we can select *two* pairs of correspondents arbitrarily in the two systems, and find one rigid body to fulfil the requirements.

If we are given two systems of the sixth order of corresponding impulsive screws and instantaneous screws then subject to the observance of the geometrical conditions, we can select *three* pairs of correspondents arbitrarily in the two systems, and find one rigid body to fulfil the requirements.

The last paragraph is, of course, only a different way of stating the results of Section IV.

XI.—TWO RIGID BODIES.

We shall now examine the circumstances under which pairs of impulsive and instantaneous screws are common to two, or more, rigid bodies. The problem before us may, perhaps, be most clearly stated as follows:—

Let there be two rigid bodies, A and B. If A be struck by an impulsive wrench on a screw θ, it will commence to twist about some screw λ. If B had been struck by an impulsive wrench on the same screw θ, the body would have commenced to twist about some screw μ, which would of course be generally different from λ. If θ be supposed to occupy different positions in space (the bodies remaining unaltered), so will λ and μ move into correspondingly various positions. It is proposed to inquire whether, under any circumstances, θ could be so placed that λ and μ should coincide. In other words, whether both of the bodies, A and B, when struck with an impulsive wrench on θ, will respond by twisting about the same instantaneous screw.

It is obvious, that there is at least one position in which θ fulfils the required condition. Let G_1 G_2 be the centres of gravity of A and B. Then a force along the ray G_1 G_2, if applied either to A or to B, will do no more than produce a linear velocity of translation parallel thereto. Hence it follows, that a wrench on the screw of zero pitch, which lies on the ray G_1 G_2, will have the same instantaneous screw whether that wrench be applied to A or to B.

We have now to examine whether there can be any other pair of impulsive and instantaneous screws in the same circumstances. Let us suppose that when θ assumes a certain position η, we have λ and μ coalescing into the single screw α.

We know that the centre of gravity lies in a plane through α, and the shortest distance between α and η. We know, also, that $d_\alpha = p_\alpha \tan \hat{\alpha\eta}$, where d_α is the distance of the centre of gravity from α. It therefore follows that α must be parallel to G_1 G_2. We have, however, already

had occasion to notice that, if ρ_a be the radius of gyration of a body about a ray through its centre of gravity, parallel to a,

$$\rho_a^2 = \frac{2p_a\varpi_{a\eta}}{\cos \hat{a\eta}} - d_a^2 - p_a^2.$$

Hence it appears that, for the required condition to be satisfied, each of the two bodies must have the same radius of gyration about the axis through its centre of gravity, which is parallel to a. Of course this will not generally be the case. It follows that, in general, there cannot be any such pair of impulsive screws and instantaneous screws, as has been supposed. Hence we have the following result:—

Two rigid bodies, with different centres of gravity, have, in general, no other common pair of impulsive screws and instantaneous screws than the screw, of zero pitch, on the ray joining the centres of gravity, and the screw of infinite pitch parallel thereto.

We shall now consider what happens when the exceptional condition, just referred to, is fulfilled, that is, when the radius of gyration of the ray $G_1 G_2$ is the same for each of the bodies.

In each of the momental ellipsoids about the centres of gravity of the two bodies, draw the plane conjugate to the ray $G_1 G_2$. Let these planes intersect in a ray, T. Suppose that an impulsive force, directed along T, be made to act on the body whose centre of gravity is G_1. It is plain, from Poinsot's well-known theory, that the rotation produced by such an impulse, will be about a ray parallel to $G_1 G_2$. If this impulsive wrench had been applied to the body whose centre of gravity is G_2, the instantaneous screw would also be parallel to $G_1 G_2$. If we now introduce the condition that the radius of gyration of each of the bodies, along $G_1 G_2$, is the same, it can be easily deduced that the two instantaneous screws are identical. Hence we see that T, regarded as an impulsive screw of zero pitch, will have the same instantaneous screw for each of the two bodies.

If we regard T and $G_1 G_2$ as two screws of zero pitch, and draw the cylindroid through these two screws, then any impulsive wrench about a screw on this cylindroid will have the same instantaneous screw to whichever of the two bodies it may be applied.

For such a wrench may be decomposed into forces on T and on $G_1 G_2$; these will produce, in either body, a twist about α, and a translation parallel to α, respectively. We therefore obtain the following theorem:—

If two rigid bodies have different centres of gravity, G_1 and G_2, and if their radii of gyration about the ray $G_1 G_2$ are equal, there is then a cylindroid of screws such that an impulsive wrench, on any one of these screws, will make either of the rigid bodies begin to twist about the same screw, and the instantaneous screws which correspond to the several screws on this cylindroid, all lie on the same ray $G_1 G_2$, but with infinitely varied pitch.

It is important to note that under no other circumstances can any impulsive screw, except the ray $G_1 G_2$, with zero pitch, have the same instantaneous screw for each of the two bodies, so long as their centres of gravity are distinct.

We might have demonstrated the theorem, above given, from the results of Section II. We have there shown that, when an impulsive screw and the corresponding instantaneous screw are given, the rigid body must fulfil five conditions, the nature of which is fully explained. If we take two bodies which comply with these conditions, it appears that the ray through their centre of gravity is parallel to the instantaneous screw, and we also found that their radii of gyration must be equal about the straight line through their centres of gravity.

If two rigid bodies have the same centre of gravity, then, of course, any ray through this point will be the seat of an impulsive wrench on a screw of zero pitch such that it generates a twist velocity on a screw of infinite pitch, parallel to the impulsive screw. This will be the case whichever of the two bodies the force be applied to. We have therefore a system of the third order (much specialized no doubt) of impulsive screws, each of which has the same instantaneous screw for each of the two bodies. In general there will be no other pairs of common impulsive and instantaneous screws beyond those indicated.

Under certain circumstances, however, there will be other screws possessing the same relation.

We may suppose the two momental ellipsoids to be drawn about the common centre of gravity. These ellipsoids will, by a well-known

property, possess one triad of common conjugate diameters. In general, of course, the radii of gyration of the two bodies, around any one of these three diameters, will not be equal. If, however, it should happen that the radius of gyration of one body be equal to that of the other body about one OX of these three common conjugate diameters, it can be shown that any screw, parallel to OX, whatever be its pitch regarded as an instantaneous screw, will have the same impulsive screw for either of the two bodies.

If the radii of gyration about two of the common conjugate diameters were equal for the two bodies, it will then appear that any instantaneous screw which is parallel to the plane of the common conjugate diameters, will have the same impulsive screw for each of the two bodies. The corresponding impulsive screws belong to the system of the fifth order, which is defined by being reciprocal to a screw of zero pitch on the third of the three common conjugate diameters.

Of course, if the radii of gyration coincide on this third diameter, then the two rigid bodies, regarded from our present point of view, would be identical.

SUMMARY OF THE TWELVE MEMOIRS.

In the spring of 1869 I happened to attend a lecture at the Royal Dublin Society, given by my friend Dr. G. Johnstone Stoney, F.R.S.

For one illustration he used a conical pendulum: he exhibited and explained the progression of the apse in the ellipse described by a heavy ball suspended from a long wire.

I was much interested by his exposition, and began immediately to work at the mathematical theory of the subject, in the endeavour to understand it more fully. I was thus led to investigate some general problems relating to the small oscillations of a particle on a surface. Certain results, at which I arrived, seemed to me interesting and novel. They appeared as a paper in the *Quarterly Journal of Pure and Applied Mathematics*, No. 39, 1869, under the title "A Problem in Mechanics."

This paper was soon followed by another on "The small oscillations of a rigid body about a fixed point under the action of any forces, and more particularly when gravity is the only force acting," *Transactions of the Royal Irish Academy*, vol. xxiv., p. 593.

The subject then began to develop into what I ventured to call the "Theory of Screws." The First Memoir with this title appeared in 1872; the Twelfth and last in 1897.

First Memoir.—"The Theory of Screws. A Geometrical Study of the Kinematics, Equilibrium, and Small Oscillations of a Rigid Body."—*Transactions of the Royal Irish Academy*, vol. xxv., pp. 157–217. Dublin, 1872.

In this paper the foundations of the theory were laid. I may here quote a passage from the Appendix to my book "Theory of Screws" (Dublin, 1876), p. 190, in which I say of this First Memoir.

"This is the original paper on the Theory of Screws. In estimating how far the contents of this paper are novel, it is to be remembered that the cylindroid had been discussed by Plücker two or three years previously, while the conception of reciprocal screws had been announced by Klein a few months before. Both these authors would, of course, have been referred to in this paper had I been acquainted with their works at the time the paper was written."

I may also quote from a foot-note, on page 16, of the same book:—

"Plücker does not appear to have contemplated the mechanical and kinematical properties of the cylindroid with which alone we are concerned; but it is worthy of remark that the distribution of pitch, which is presented by physical considerations, is exactly the same as the distribution of parameter upon the generators of the surface, which was fully discussed by Plücker in connexion with his theory of the linear complex."

I learned from Professor Klein that Plücker had made a model of this surface. The first model I made was presented to the Mathematical Society of London. A subsequent model, also made by myself, is the

property of the Royal College of Science, in Dublin. This model is figured in a plate of the First Memoir, and is also given in the frontispiece to the "Theory of Screws," 1876.

A most beautiful model was made for me, about 1883, by my friend Sir Howard Grubb. A stereoscopic view of this forms the frontispiece in Gravelius' work, to be presently mentioned. This model shows the nodal line. The three cuspidal edges on the tangent cone, from any point to the surface, are also delicately indicated. Sir Howard Grubb made another model, with certain modifications, for my friend Mr. G. L. Cathcart, Fellow of Trinity College, Dublin.

Second Memoir.—"Researches in the Dynamics of a Rigid Body by the aid of the Theory of Screws."—*Philosophical Transactions of the Royal Society*, London, 1874, pp. 15–40.

The chief feature in this paper is the general theorem that a rigid body has just so many principal screws of inertia as it has degrees of freedom. This principle is a generalization for all cases of a rigid system, no matter what be the nature and number of its constraints, of the well-known property of the principal axes of a rigid body rotating around a fixed point.

The discovery of the principal screws of inertia marked a fundamental step in the theory. It has led to many of the subsequent developments. The geometrical construction of these screws is specially considered in the Twelfth Memoir.

Another department in the theory which takes its rise from the Second Memoir is the perception of the dynamical significance of homographic systems of screws. In this Memoir it is shown that if the screws on one cylindroid be regarded as impulsive screws, the system of corresponding instantaneous screws lie on another cylindroid. Any four screws on the one cylindroid, and their four correspondents on the others are equi-anharmonic. This theorem leads to many points of connexion between theoretical

dynamics and modern geometry. It has been greatly developed in subsequent Memoirs.

A postscript to this paper gives a brief historical sketch which shows the relation of the theory of screws to the classical researches of Plücker and Klein on the "Theory of the Linear Complex."

Third Memoir.—"Screw Coordinates and their Applications to Problems in the Dynamics of a Rigid Body."—*Transactions of the Royal Irish Academy*, vol. xxv., pp. 259–327 (1874).

The progress of the present theory was much facilitated by the introduction of screw-coordinates. The origin and the use of such coordinates are explained in this Memoir. It is, however, to be understood that screw coordinates are an adaptation for dynamical purposes of Klein's coordinates of a linear complex referred to six fundamental complexes, of which each pair are in involution (or reciprocal, as we say in the "Theory of Screws").

The pitch of a screw α as expressed in terms of its six coordinates $\alpha_1, \alpha_2, \ldots \alpha_6$ is $\Sigma p_1 \alpha_1^2$ where $p_1 \ldots p_6$, &c. are the pitches of the coreciprocal screws of reference. The virtual coefficient of two screws α and β is $\Sigma p_1 \alpha_1 \beta_1$. In the dynamical part of the subject the chief result of this paper is the theorem that, when the six screws of reference are the six principal screws of inertia, then $p_1\alpha_1, p_2\alpha_2, \ldots p_6\alpha_6$ are the coordinates of the impulsive wrench which will make the body commence to move by twisting about the screw $\alpha_1 \ldots \alpha_6$.

This was, perhaps, all that could be desired in the way of a simple connexion between an impulsive screw and the corresponding instantaneous screw, so far as their coordinates were concerned. Long before this paper was published I had been trying to find the geometrical connexion between the screws which would exhibit their relation in a graphic manner. But the search was not to be successful until the results in the Twelfth Memoir were arrived at.

The results of the first three Memoirs were incorporated in a volume

which I published in 1876 under the title "The Theory of Screws, a Study in the Dynamics of a Rigid Body," 194 pp. (Dublin: Hodges, Foster, & Co.). The appendix, pp. 177–192, describes the principal works which were then known to me bearing on the subject of the volume.

FOURTH MEMOIR.—"Extension of the Theory of Screws to the Dynamics of any Material System."—*Transactions of the Royal Irish Academy*, vol. xxviii., pp. 99–136 (1881).

In this paper I show a method by which the principles of the Theory of Screws may be employed to give graphic representations of certain problems relating to the dynamics of any material system whatever.

The conception of a screw-chain is here introduced. The screw-chain is a geometrical entity which bears to an entire system, no matter how complex its parts or their connexions, the same relation which a screw bears to a single rigid body. One screw-chain can always be found which is reciprocal to $6\mu - 1$ screw-chains where μ is the number of material parts in the system.

One of the chief results obtained shows the extension of the notion of the principal screws of inertia of a single rigid body to a system of rigid bodies. The following general theorem is demonstrated :—

If any mechanical system whatever have n degrees of freedom, then n screw-chains can always be found (but not generally more than n) such that if the system be disturbed from rest by an impulsive wrench on any one of these screw-chains, it will immediately commence to move by twisting about the same screw-chain.

The theory of the small oscillations of a material system about a position of equilibrium can be also illustrated by the conception of the screw-chains. It is shown that there are n harmonic screw-chains, such that if the system be slightly displaced from a position of stable equilibrium by a twist about one of these screw-chains, and be then released, it will continue for ever to twist to and fro on the same screw-chain.

When the results of the Twelfth Memoir are incorporated with those of

the Fourth we obtain a geometrical theory of equilibrium of impulsive forces and of small oscillation for a material system of the most general type.

Fifth Memoir.— 'Certain Problems in the Dynamics of a Rigid System moving in Elliptic Space."—*Transactions of the Royal Irish Academy*, vol. xxviii., pp. 159–184 (1881).

This Memoir discusses the Theory of Screws in Elliptic space. The work is based on Clifford's remarkable papers on Biquaternions, and, of course, the well-known labours of Cayley, Klein, and Lindemann are referred to.

The chief theorem proved in this paper is, that though the virtual moment of two homonymous vectors is zero only when the two vectors are "rectangular," yet the virtual moment of two heteronymous vectors is always zero.

I may here mention another Memoir which bears on the subject of elliptic space, though as it relates to other matters besides the Theory of Screws, I have not enumerated it as one of the Twelve Memoirs of the series of which I am now giving the Summary. The title is, "On the Theory of the Content"—*Transactions of the Royal Irish Academy*, vol. xxix., pp. 123–181.

In this it is shown, *inter alia*, that the order in which two heteronymous vectors in elliptic space are applied to a rigid system may be inverted without affecting the result which is, however, not a vector at all. On the other hand, when two homonymous vectors in elliptic space are applied to a rigid system, the result is, in every case, a homonymous vector; but then the order of application could not be inverted without changing the result. The theory of the dynamics of a rigid system in elliptic space must, it would seem, be largely dependent on this theorem, together with that just referred to in the Fifth Memoir.

Professor Heath's beautiful theorem in elliptic space is made use of.

Sixth Memoir.—"Dynamics and Modern Geometry: a new chapter in the Theory of Screws."—*Cunningham Memoirs of the Royal Irish Academy*, No. IV., pp. 1–44 (1886).

Seeing the paramount importance of the cylindroid in the present theory, it is clearly an advantage to have a simple method by which this surface can be represented in a plane. We here give such a method by which the dynamical problems connected with this ruled cubic surface can be solved by elementary plane geometry.

We represent the several screws on the cylindroid by points on the circumference of a circle. The angle between two screws is the angle which their corresponding points subtends at the circumference. The shortest distance of any two screws is the projection of the corresponding chord on a fixed ray in the plane of the circle. Any chord passing through the pole of this ray intersects the circle in points corresponding to reciprocal screws. The pitch of any screw is the distance of its corresponding point from this ray. A system of points representing instantaneous screws and the corresponding system representing the impulsive screws are homographic. The double points of the homography correspond to the two principal screws of inertia.

Seventh Memoir.—"On the Plane Sections of the Cylindroid."—*Trans. of the Royal Irish Academy*, vol. xxix., pp. 1–32 (1887).

This is a geometrical study of the cylindroid regarded as a conoidal cubic with one nodal line and three right lines in the plane at infinity. Plane sections of the cylindroid are shown in plates drawn to illustrate calculated cases.

Various theorems are demonstrated. It is shown, for example, that the chord joining the points in which two reciprocal screws intersect a fixed plane envelops an hyperbola which has triple contact with the cubic curve in which the fixed plane cuts the cylindroid.

There is but little in this paper relating to the dynamical development of the subject. A reader of the subsequent Memoirs would not find any difficulty to arise because he had omitted this one.

Eighth Memoir.—"How Plane Geometry illustrates general problems in the Dynamics of a Rigid Body with three degrees of Freedom."—*Transactions of the Royal Irish Academy*, vol. xxix., pp. 247–284.

The system of the third order is of such special interest that it is very desirable to have a concise method of representing the screws which constitute it. We here show that the screws of such a system correspond to the points in a plane.

In this method of representation the screws on a cylindroid belonging to the system are represented by the points on a straight line. The screws of any given pitch will have as their correspondents the points on a certain conic. A pair of points conjugate to the conic of zero pitch will correspond to a pair of reciprocal screws. The conic which represents the screws of zero pitch, and the conic which represents the screws of infinite pitch, will have a common conjugate triangle. The vertices of that triangle correspond to the principal screws of the system.

The points which represent a series of impulsive screws, and the points which represent the series of corresponding instantaneous screws are homographic. The three double points of the homography represent the three principal screws of inertia.

The three harmonic screws about any one of which the body would oscillate for ever in the vicinity of a position of stable equilibrium are determined as the vertices of the common conjugate triangle of two conics.

A very complete account in German of the "Theory of Screws," 1876, and of the subsequent Memoirs down to and including the eighth, as well as several other shorter papers not mentioned in this summary, has been given in a treatise by Herr Gravelius. The work has been enriched by the original investigations made by the author, especially in Chap. XIX. The complete reference is "Theoretische Mechanik Starrer Systeme auf Grund der Methoden und Arbeiten und mit einem Vorworte von Sir Robert S. Ball." Herausgegeben von Harry Gravelius, 8vo, pp. 1–619. Georg Reimer, Berlin, 1889. I would like to mention here the highly suggestive review of this book by Professor Henrici—*Nature*, June 5, 1890.

NINTH MEMOIR.—"The Theory of Permanent Screws."—*Transactions of the Royal Irish Academy*, vol. xxix., pp. 613–652.

"In the ordinary theory of the principal axes of a rigid body there are, as is well known, two distinct properties of such an axis which possess dynamical significance. We may think of a principal axis as the axis of a couple which, when applied impulsively to the body, will set it spinning about this line. We may also think of the principal axes as a direction about which, if a body be once set in rotation, it will continue to rotate. When the first of these properties is suitably generalized it opens up the theory of the principal screw-chains of inertia which I have already explained in previous Memoirs. It is from the other property of the principal axis that the present theory takes its rise. The two general theories are wholly distinct in their mathematical treatment, and in the character of the conceptions to which they give rise. It is, therefore, to be observed that two different departments in the Theory of Screws happen to coalesce in the very special case of a rigid body rotating around a point."

I have occasion to demonstrate a general property of the function T which expresses the kinetic energy. If expressed in screw-chain coordinates this property is

$$\dot{x}_1 \frac{dT}{dx_1} + \ldots + \dot{x}_n \frac{dT}{dx_n} = 0.$$

It is shown that for the permanent screw-chains,

$$\frac{dT}{dx_1} = 0 \ldots \frac{dT}{dx_n} = 0.$$

The special cases for the different degrees of freedom of a single rigid body are considered in detail. If the rigid body has three degrees of freedom then there are three permanent screws, about any one of which the body will continue to twist if once set twisting.

In general, if the body be set twisting about a screw θ, a restraining wrench on some other screw η would be necessary if the motion were to continue as a twist about θ.

To find η we employ the plane representation. We construct first a system of points homographic with the points θ. The double points of the homography are representative of the three permanent screws. If we draw the ray connecting θ with its correspondent, then η is the pole of this ray with respect to the conic of zero pitch, while the pole of the same ray with respect to the conic of inertia gives the screw about which the acceleration is imparted to θ.

For freedom of the first and second orders there is only one permanent screw; for freedom of the third, fourth, and fifth, there are three permanent screws. When the body is quite free the permanent screws are triply infinite.

TENTH MEMOIR.—"The Theory of Pitch Invariants and the Theory of Chiastic Homography."—*Transactions of the Royal Irish Academy*, vol. xxx., pp. 559–586.

It is shown that if $a_1 \ldots a_6$ be the six coordinates of a screw α, while $h_1, \ldots h_6$ are the angles which α makes with the six coreciprocal screws of reference, then expressions of the form

$$a_1 \cos h_1 + \ldots + a_6 \cos h_6$$

are invariants in the sense that they are unaltered for every screw on the same ray as α.

If $k_1, \ldots k_6$ be the similar angles for any other screw, then

$$\frac{\cos h_1 \cos k_1}{p_1} + \frac{\cos h_2 \cos k_2}{p_2} + \ldots + \frac{\cos h_6 \cos k_6}{p_6} = 0,$$

where $p, \ldots p_6$ are the pitches of the screws of reference.

In the first Memoir I had explained the property of conjugate screws of inertia. If two instantaneous screws α and β and the corresponding impulsive screws η and ξ were so related that α was reciprocal to ξ, then β must be reciprocal to η. This clearly implied that there must be some relation in all cases between the virtual coefficients $\varpi_{\alpha\xi}$ and $\varpi_{\beta\eta}$. But I

knew this long before I learned what the relation was. It is here shown to be

$$\frac{p_\alpha}{\cos \hat{\alpha\eta}} \varpi_{\beta\eta} = \frac{p_\beta}{\cos \hat{\beta\xi}} \varpi_{\alpha\xi}.$$

In this paper also the notion of chiastic homography is introduced. The characteristic feature of chiastic homography is, that every three pairs of correspondents α, η; β, ξ; γ, ζ, fulfil the relation

$$\varpi_{\alpha\xi} \varpi_{\beta\zeta} \varpi_{\gamma\eta} = \varpi_{\alpha\zeta} \varpi_{\beta\eta} \varpi_{\gamma\xi}.$$

The homography of impulsive and instantaneous systems is chiastic, and the relation has other physical applications.

ELEVENTH MEMOIR.—"Further Development of the Relations between Impulsive Screws and Instantaneous Screws." —*Transactions of the Royal Irish Academy*, vol. xxxi., pp. 99–144.

It is shown that when η is the impulsive screw and α the instantaneous screw, the kinetic energy of mass M twisting about α with a twist velocity $\dot{\alpha}$ is

$$M\dot{\alpha}^2 \frac{p_\alpha}{\cos \hat{\alpha\eta}} \varpi_{\alpha\eta}.$$

The twist velocity acquired by a given impulse is proportional to

$$\frac{\cos \hat{\alpha\eta}}{p_\alpha}.$$

There is a second general relation, besides that proved in the last Memoir, between two pairs of impulsive screws η, ξ, and their corresponding instantaneous screws α, β for a free body

$$\frac{p_\alpha}{\cos \hat{\alpha\eta}} \cos \hat{\beta\eta} + \frac{p_\beta}{\cos \hat{\beta\xi}} \cos \hat{\alpha\xi} = 2\varpi_\alpha .$$

The following instructive theorem, of which much use is made in the next Memoir, has been proved in this one.

If two cylindroids be given there is, in general, one, and only one, possible correlation of the screws on the two surfaces, such that a rigid body could be constructed for which the screws on one cylindroid would be the impulsive screws, and their correspondents on the other cylindroid the instantaneous screws.

I have also shown that when three pairs of corresponding impulsive screws and instantaneous screws were known, all other such pairs are in general determined. The fuller geometrical development of this fundamental chapter in Rigid Dynamics is, however, reserved for the Twelfth and last Memoir.

Twelfth Memoir.—"The Twelfth and concluding Memoir on the Theory of Screws, with a Summary of the Twelve Memoirs."—*Transactions of the Royal Irish Academy*, vol. xxxi., pp. 145–196.

At last I succeeded in accomplishing what I had attempted from the first. I could not develop the complete theory, which I felt certain must exist, until I had obtained a geometrical method for finding the instantaneous screw from the impulsive screw. When this was accomplished in the midsummer of 1897, the geometrical theory in Dynamics, which I had striven for a quarter of a century to obtain, was at last manifest. How my difficulties were overcome has been set forth in this Memoir.

There are, no doubt, many other questions that would repay investigation at almost every stage of the subject. But the problem which I had proposed to myself so many years ago, and which I have steadily kept in view ever since, having been at last resolved, I have felt that this series of papers should cease.

THE END.

APPENDIX TO PART V.

AMENDMENT TO "THE TWELFTH AND CONCLUDING MEMOIR ON THE THEORY OF SCREWS": p. 172 *supra*. By SIR ROBERT S. BALL, LL.D., F.R.S., Lowndean Professor of Astronomy and Geometry in the University of Cambridge.

[Read April 25th, 1898.]

I had occasion to set down, on page 172 of the above-mentioned Paper, a construction for the three double points of two plane homographic systems. It is no doubt true that the three double points possess the property there stated, but the converse does not generally hold, as I had incautiously supposed. The construction given is therefore unsound.

A correct construction for the double points of two homographic systems of points in the same plane is as follows:—

Let O and O' be a pair of corresponding points. Then each ray through O will have, as its correspondent, a ray through O'. The locus of the intersection of these rays will be a conic S. This conic S must pass through the three double points, and also through O and O'.

Draw the conic S', which is the locus of the points in the second system corresponding to the points on S, regarded as in the first system. Then since O lies on S, we must have O' on S'. But S' must also pass through the three double points. O' is one of the four intersections of S and S', and the three others are the sought double points. Thus the double points are constructed.

On the same page (*loc. cit.*) I also set down a construction for the four double points of two homographic systems in space. Here again the process is not generally applicable. I replace it by a correct construction, as follows:—

Let O and O' be two corresponding rays. Then any plane through O will have, as its correspondent, a plane through O'. It is easily seen

that these planes intersect on a ray which has for its locus a quadric surface S, of which O and O' are also generators. This quadric must pass through the four double points.

Let S' be the quadric surface which contains all the points in the second system corresponding to the points of S regarded as the first system. Then O' will lie on S', and the rest of the intersection of S and S' will be a twisted cubic C, which passes through the four double points.

Take any point P on C, and draw any plane through P. Then every ray of the first system of the pencil through P in this plane will have as its correspondent in the second system the ray in some other plane pencil L. One, at least, of the rays in the pencil L will cut the cubic C. Call this ray X', and draw its correspondent X in the first system passing through P.

We thus have a pair of corresponding rays X and X', each of which intersects the twisted cubic C.

Draw pairs of corresponding planes through X and X'. The locus of their intersection will be a quadric S'', which also contains the four double points.

S'' and C, being of the second and the third order respectively, will intersect in six points. Two of these are on X and X', and are thus distinguished. The four remaining intersections will be the required double points, and thus the problem has been solved.